A NOTE ON THE ZERO COUPON BOND PRICING USING MERTON'S NONLINEAR MEAN REVERSION INTEREST RATE MODEL

ANTONIO FALCÓ AND JUAN M. NAVE

ABSTRACT. The aim of this paper is to obtain a closed formula for a zero coupon $T-$ bond where the interest rate follows a Nonlinear Mean Reversion Model given by Merton in 1975. It is well-known that this model has no closed–form transition density. However, we will prove that the Partial Differential Equation governing the arbitrage–free bond values has self–similar solutions. By using this fact, we will reduce the Partial Differential Equation down to the well-known Kummer's Differential Equation. In consequence, the arbitrage–free price of a zero coupon $T-$bond can be expressed by means of a Confluent Hypergeometric Function.

1. Introduction and statement of the main result

In this note we will consider the short–term interest rate r introduced by Merton (1975) (see also Merton (1990), Chapter 17). This model follows an Itô process, under the actual probability $\mathbb{P}$, defined by the following Stochastic Differential Equation

$$dr(t) \quad = \quad r(t)\left(a - br(t)\right)dt + \sigma r(t)dW(t), \;\; r(0) = r, \tag{1}$$

where $a, b, \sigma \in \mathbb{R}^+$. From Itô's Lemma it can be shown that

$$r(t) = \frac{r\exp\left(\left(a - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right)}{1 + rb\int_0^t \exp\left(\left(a - \frac{1}{2}\sigma^2\right)s + \sigma W(s)\right)ds}. \tag{2}$$

Thus if $r > 0$, then $r(t) > 0$, with probability one. Otherwise, if $r = 0$, then $r(t) = 0$, with probability one. Moreover, when $r > 0$, we have that $r(t)$ has a Gamma distribution as $t \to \infty$ (see Merton (1975)). It is well-known that this model has no closed–form transition density. Moreover, it can be included in the class of Nonlinear Mean Reversion

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Models (see Aït–Sahalia (1999)) given by

$$dr(t) = \left(a_{-1}r(t)^{-1} + a_0 + a_1 r(t) + a_2 r^2(t)\right) dt + \sigma r^{\varrho}(t) dW(t)$$

when

$$a_{-1} = 0,\ a_0 = 0,\ a_1 = a,\ a_2 = b \text{ and } \varrho = 1.$$

Let $p = p(t, r; T)$ denote the price of a zero coupon $T-$bond. Now, in order to obtain the Term Structure Equation, we assume that the bond market is arbitrage–free. Then, it follows (see Björk (1998)) that there exists a universal process $\lambda(t)$, called the risk premium or the market price of risk, such that

$$p_t + \frac{1}{2}\sigma^2 r^2 p_{rr} + r\,(a - \sigma\lambda - br)\,p_r - rp = 0, \tag{3}$$

with boundary condition

$$p(T, r; T) = 1.$$

From the Feynman–Kac Representation Theorem we have that

$$p(t, r; T) = E_{\mathbb{P}^{\lambda}}^{(t,r)}\left[\exp\left(-\int_t^T r(s)ds\right)\right], \tag{4}$$

where $r(t)$ follows under this risk–neutral measure,

$$dr(t) = r(t)\,(a - \sigma\lambda(t) - br(t))\,dt + \sigma r(t) dW^{\lambda}(t),\ r(0) = r. \tag{5}$$

We note that

$$p(t, 0; T) = 1,$$

for all $t \geq 0$.

In the case where the transition function is unknown, in order to obtain a solution in closed form we must solve the Partial Differential Equation (3). Otherwise, one must rely on numerical methods such as solving the Partial Differential Equation satisfied by the $T-$bond price numerically, or Montecarlo integration of equation (4). Recently, Lewis (1998) has obtained a solution of equation (3) by using eigenfunctions expansions. Unfortunately, it has a very complicated expression.

In this paper we wish to find a closed solution of the $T-$bond price in similar terms as the well-known Black–Scholes formula. To obtain this, we will show that the Partial Differential Equation associated with the bond price has a symmetry. A symmetry is a transformation of p, r, t, a, b, σ^2 and λ which leaves the underlying Partial Differential Equation unchanged. The general theory of this transformation can be found in Olver (1986) and Dresner (1983).

More precisely, we will use a symmetry which belongs to an important sub–class of symmetries which are given by a scaling transformation. These scaling symmetries are universal in physics. They can be found, for example, in fluid mechanics, turbulence, elasticity and mathematical biology. Scaling invariance is also, of course, closely tied up with the Renormalization Group Theory (see Barrenblatt (1996)).

Solutions which are therefore invariant for a scaling transformation are called self–similar solutions. A self–similar solution also satisfies an equation simpler than the underlying Partial Differential Equation. Indeed they are often satisfy an Ordinary Differential Equation.

By using this approach we will show that equation (3) has self–similar solutions. From this fact we will reduce the Partial Differential Equation to the Kummer's Differential Equation (see Abramowitz and Stegun (1992)). Now, before state our main result we introduce the following Confluent Hypergeometric Functions.

A function $M(\alpha,\beta,z)$ is called a *Kummer function of the first order* if

$$M(\alpha,\beta,z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\beta)_n}\frac{z^n}{n!},$$

where

$$(\alpha)_n=\alpha(\alpha+1)\cdots(\alpha+n-1)=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},$$

and $(\alpha)_0=1$. These functions are well-defined for all values of α,β and z, if $1-\alpha, 1-\beta\notin\mathbb{Z}^+$. Moreover, M is the unique solution of the so–called Kummer's Differential Equation given by

$$z\frac{d^2M(\alpha,\beta,z)}{dz^2}+(\beta-z)\frac{dM(\alpha,\beta,z)}{dz}-\alpha M(\alpha,\beta,z)=0,$$

with the boundary condition

$$M(\alpha,\beta,0)=1.$$

Then the main result of this paper is the following.

Theorem A. *Assume that $r(t)$ follows the Merton's Nonlinear Mean Reversion Interest Rate Model (1) and let*

$$\theta=\frac{(a-\sigma\lambda)(T-t)-1}{\frac{1}{2}\sigma^2(T-t)}.$$

If $\lambda(t)=\lambda$ for some real number $\lambda>0$ and $1-\theta\notin\mathbb{Z}^+$. Then the non-arbitrage price of a zero coupon $T-$bond is given by

$$p(t,r;T)=M\left(b^{-1},\theta,2br/\sigma^2\right).$$

Moreover, $p(t,r;T)$ is non-increasing and convex as a function of the initial rate r, and if $1-2(a-\sigma\lambda)/\sigma^2\notin\mathbb{Z}^+$, then

$$\lim_{(T-t)\to\infty}p(t,r;T)=M\left(b^{-1},2(a-\sigma\lambda)/\sigma^2,2br/\sigma^2\right).$$

The rest of the paper is devoted to the proof of Theorem A.

2. Proof of Theorem A

First at all we fix $T > 0$. Let

$$u(t, r, a, b, \sigma^2, \lambda).$$

be the solution of the following Partial Differential Equation,

$$u_t + \frac{1}{2}\sigma^2 r^2 u_{rr} + r\,(a - \sigma\lambda - br)\,u_r - ru = 0, \tag{6}$$

with boundary condition

$$u(T, r, a, b, \sigma^2, \lambda) = 1. \tag{7}$$

Let $\tau = T - t$, be the time to maturity and take $\mu = a - \sigma\lambda$. Then we can write (6) as

$$-u_\tau + \frac{1}{2}\sigma^2 r^2 u_{rr} + r\,(\mu - br)\,u_r - ru = 0, \tag{8}$$

with boundary condition

$$u(0, r, a, b, \sigma^2, \lambda) = 1. \tag{9}$$

By using the Method of Similarity solutions for Partial Differential Equations it is easy to see that

$$u(\tau, r, a, b, \sigma^2, \lambda) = u(\overline{\tau}, \overline{r}, \overline{a}, b, \overline{\sigma}^2, \overline{\lambda})$$

where

$$\tau = \bar{\tau} L^{\alpha},\ r = \bar{r} L^{-\alpha},\ a = \overline{a} L^{-\alpha},\ \sigma^2 = \overline{\sigma}^2 L^{-\alpha},\ \lambda = \overline{\lambda} L^{-\alpha/2}$$

for $L > 0$ and α in $\mathbb{R}$. Then, taking $\tau = L^{\alpha}$ we obtain that

$$u(\tau, r, a, b, \sigma^2, \lambda) = u(1, r\tau, a\tau, b, \sigma^2\tau, \lambda\tau^{1/2}) \triangleq f(r\tau).$$

Let $x = r\tau$, then

$$\begin{aligned} u_r &= \tau f'(x), \\ u_{rr} &= \tau^2 f''(x), \\ u_\tau &= r f'(x). \end{aligned}$$

Substituting in (8) we obtain that

$$\frac{1}{2}\sigma^2 r^2 \tau^2 f''(x) + r(\mu - br)\tau f'(x) - r f'(x) - r f(x) = 0, \tag{10}$$

that is,

$$r\left(\frac{1}{2}\sigma^2 \tau x f''(x) + ((\mu\tau - 1 - bx)\,f'(x) - f(x)\right) = 0. \tag{11}$$

for all $r \geq 0$ and $\tau \geq 0$. Thus, we need to solve

$$\frac{1}{2}\sigma^2 \tau x f''(x) + ((\mu\tau - 1 - bx)\,f'(x) - f(x) = 0. \tag{12}$$

with boundary condition

$$u(0, r, a, b, \sigma^2, \lambda) = u(t, 0, a, b, \sigma^2, \lambda) = f(0) = 1.$$

Assume that $\tau > 0$. Then, if we denote by

$$\theta = \frac{(\mu\tau - 1)}{\frac{1}{2}\sigma^2\tau}, \;\; \rho = \frac{1}{\frac{1}{2}\sigma^2\tau},$$

we can write equation (12) as

$$xf''(x) + (\theta - b\rho x)\, f'(x) - \rho f(x) = 0, \tag{13}$$

with boundary condition $f(0) = 1$. We note that the parameters θ and ρ do not contain L. Moreover,

$$\rho x = \frac{2r}{\sigma^2}. \tag{14}$$

Now, consider the change of variable

$$z = b\rho x = \frac{2br}{\sigma^2}$$

and take

$$f(x) = w(z).$$

Then, w solves the Kummer's Differential Equation

$$zw''(z) + (\theta - z)w'(z) - b^{-1}w(z) = 0 \tag{15}$$

with boundary condition

$$w(0) = 1.$$

In consequence, if $1 - \theta \notin \mathbb{Z}^+$, then

$$w(z) = M(b^{-1}, \theta, z) = M\left(b^{-1}, \theta, 2br/\sigma^2\right).$$

Thus, by using (14), the solution of the Partial Differential Equation (8), for $r \geq 0$ and $\tau > 0$, is

$$u(r, \tau, a, b, \sigma^2, \lambda) = M\left(b^{-1}, \theta, 2br/\sigma^2\right).$$

We remark, that

$$\frac{(b^{-1})_n}{(\theta)_n} = \left(\frac{1}{2}\sigma^2\tau\right)^n \frac{(b^{-1})_n}{\prod_{j=0}^{n-1}\left((a - \sigma\lambda)\tau - 1 - j(1/2)\sigma^2\tau\right)}$$

for all $n \geq 1$. This last equality implies that

$$\lim_{\tau\to 0} \frac{(b^{-1})_n}{(\theta)_n} = 0,$$

for all $n \geq 1$. In consequence,

$$\lim_{\tau\to 0} u(r, \tau, a, b, \sigma^2, \lambda) = \lim_{\tau\to 0} M\left(b^{-1}, \theta, 2br/\sigma^2\right) = 1.$$

This follows the first statement of Theorem A.

To prove the second one, we use Theorem 2 of Alvarez (2001). This result implies that, for the Merton's Nonlinear Mean Reversion Interest Rate Model, the price $p(t, r; T)$ of a zero coupon $T-$bond is non-increasing and convex as a function of the initial rate r.

Finally, recall that

$$\theta = \frac{(a-\sigma\lambda)\tau - 1}{\frac{1}{2}\sigma^2\tau},$$

and

$$u(\tau, r, a, b, \sigma^2, \lambda) = f(x) = w(z) = w\left(2br/\sigma^2\right).$$

By taking limits in (15) when $\tau \to \infty$, we obtain that w solves

$$zw''(z) + \left(\frac{2(a-\sigma\lambda)}{\sigma^2} - z\right)w'(z) - b^{-1}w(z) = 0,$$

with boundary condition $w(0) = 1$. Thus, if $1 - 2(a-\sigma\lambda)/\sigma^2 \notin \mathbb{Z}^+$, then

$$\lim_{(T-t)\to\infty} p(t, r; T) = w(z) = M\left(b^{-1}, 2(a-\sigma\lambda)/\sigma^2, 2br/\sigma^2\right).$$

This ends the proof of Theorem A.

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Departamento de Economía y Empresa, Facultad de Ciencias Sociales Jurídicas, Campus de Elche, Universidad Cardenal Herrera-CEU, Comissari 1, 03203 Elche-Elx, Spain

E-mail address: afalco@uch.ceu.es

Departamento de Economía y Empresa, Universidad de Castilla la Mancha, Facultad de Ciencias Sociales. Avda. Alfares, 44. E-16071 Cuenca, Spain

E-mail address: Juan.Nave@uclm.es